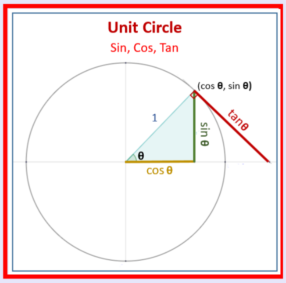
Math 241

Winter 2024

Lecture 8



More identities:

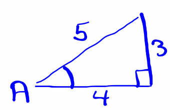
$\sin(A+B) = \sin A \cos B + \cos A \sin B$

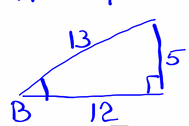
$\cos(A+B) = \cos A \cos B - \sin A \sin B$

$\tan(A+B) = \ ?$

ex: $0^\circ < A < 90^\circ$, $\sin A = \frac{3}{5}$

$0^\circ < B < 90^\circ$, $\cos B = \frac{12}{13}$





$\sin(A+B) = \sin A \cos B + \cos A \sin B$

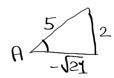
$= \frac{3}{5} \cdot \frac{12}{13} + \frac{4}{5} \cdot \frac{5}{13} = \frac{36}{65} + \frac{20}{65} = \boxed{\frac{56}{65}}$

$\cos(A+B) = \cos A \cos B - \sin A \sin B$

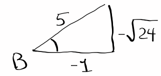
$= \frac{4}{5} \cdot \frac{12}{13} - \frac{3}{5} \cdot \frac{5}{13} = \frac{48}{65} - \frac{15}{65} = \boxed{\frac{33}{65}}$

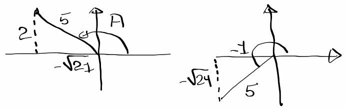
$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{56/65}{33/65} = \boxed{\frac{56}{33}}$

Given $90° < A < 180°$, $\sin A = \frac{2}{5}$
QII



$180° < B < 270°$, $\cos B = \frac{-1}{5}$
QIII





$\sqrt{21} = \sqrt{3}\sqrt{7}$

$\sqrt{24} = \sqrt{3}\sqrt{4}\sqrt{2}$

$= 3 \cdot 2\sqrt{14}$

$\sin(A+B) = \sin A \cos B + \cos A \sin B$

$= \frac{2}{5} \cdot \frac{-1}{5} + \frac{-\sqrt{21}}{5} \cdot \frac{-\sqrt{24}}{5} = \boxed{\frac{-2+6\sqrt{14}}{25}}$

$\sqrt{24} = \sqrt{4}\sqrt{6}$

$\cos(A+B) = \cos A \cos B - \sin A \sin B$

$= \frac{-\sqrt{21}}{5} \cdot \frac{-1}{5} - \frac{2}{5} \cdot \frac{-\sqrt{24}}{5} = \frac{\sqrt{21} + 2 \cdot 2\sqrt{6}}{25}$

$= \boxed{\frac{\sqrt{21} + 4\sqrt{6}}{25}}$

$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\frac{-2+6\sqrt{14}}{\cancel{25}}}{\frac{\sqrt{21}+4\sqrt{6}}{\cancel{25}}} = \frac{-2+6\sqrt{14}}{\sqrt{21}+4\sqrt{6}} \cdot \frac{\sqrt{21}-4\sqrt{6}}{\sqrt{21}-4\sqrt{6}}$

$= \frac{-2\sqrt{21} + 8\sqrt{6} + 7\sqrt{6} - 48\sqrt{21}}{(\sqrt{21})^2 - (4\sqrt{6})^2} = \frac{-50\sqrt{21} + 15\sqrt{6}}{21 - 166} = \frac{-5(10\sqrt{21} - 3\sqrt{6})}{-75}$

$\sqrt{14}\sqrt{21} = \cancel{\sqrt{7}}\sqrt{2}\cancel{\sqrt{7}}\sqrt{3}$
$= 7\sqrt{6}$

$\sqrt{14}\sqrt{6} = \cancel{\sqrt{2}}\sqrt{7}\cancel{\sqrt{2}}\sqrt{3}$
$= 2\sqrt{21}$

$= \boxed{\frac{10\sqrt{21} - 3\sqrt{6}}{15}}$

Derive a formula for $\tan(A+B)$

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$

Divide everything by $\cos A \cos B$, and Simplify

$$= \frac{\frac{\sin A \cancel{\cos B}}{\cos A \cancel{\cos B}} + \frac{\cancel{\cos A} \sin B}{\cancel{\cos A} \cos B}}{\underbrace{\frac{\cancel{\cos A \cos B}}{\cancel{\cos A \cos B}}}_{1} - \frac{\sin A \sin B}{\cos A \cos B}}$$

$$\tan(A+B) = \boxed{\frac{\tan A + \tan B}{1 - \tan A \tan B}}$$

Find exact Value for $\sin 75°$, $\cos 75°$, and $\tan 75°$

$75° = 30° + 45°$

$\sin 75° = \sin(30° + 45°) = \sin 30° \cos 45° + \cos 30° \sin 45°$

$A + B$ $\sin A \cos B + \cos A \sin B$

$= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} = \boxed{\frac{\sqrt{2} + \sqrt{6}}{4}}$

$\cos 75° = \cos(30° + 45°) = \cos 30° \cos 45° - \sin 30° \sin 45°$

$= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{1}{2} \cdot \frac{\sqrt{2}}{2} = \boxed{\frac{\sqrt{6} - \sqrt{2}}{4}}$

$\tan 75° = \tan(45° + 30°) = \frac{\tan 45° + \tan 30°}{1 - \tan 45° \cdot \tan 30°}$

$A + B$

$= \frac{1 + \frac{\sqrt{3}}{3}}{1 - 1 \cdot \frac{\sqrt{3}}{3}} = \frac{3 + \sqrt{3}}{3 - \sqrt{3}}$

$= \frac{3+\sqrt{3}}{3-\sqrt{3}} \cdot \frac{3+\sqrt{3}}{3+\sqrt{3}} = \frac{9 + 3\sqrt{3} + 3\sqrt{3} + 3}{(3)^2 - (\sqrt{3})^2} = \frac{12 + 6\sqrt{3}}{9 - 3}$

$= \frac{\cancel{6}(2 + \sqrt{3})}{\cancel{6}} = \boxed{2 + \sqrt{3}}$ $\rightarrow \approx 3.732$

Use your calc.
$\tan 75° \approx 3.732$

Find exact value for $\sin 105°$, $\cos 105°$, and $\tan 105°$.

Hint: $105° = 60° + 45°$

$\sin 105° = \sin(60° + 45°) = \sin 60° \cos 45° + \cos 60° \sin 45°$

$A + B$ $= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} + \frac{1}{2} \cdot \frac{\sqrt{2}}{2}$

$= \boxed{\frac{\sqrt{6} + \sqrt{2}}{4}}$

$\cos 105° = \cos(60° + 45°) = \cos 60° \cos 45° - \sin 60° \sin 45°$

$A + B$

$= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2}$

$= \boxed{\frac{\sqrt{2} - \sqrt{6}}{4}}$

$\tan 105°$

Method I: $\tan 105° = \dfrac{\sin 105°}{\cos 105°}$

$$= \frac{\frac{\sqrt{6}+\sqrt{2}}{\cancel{4}}}{\frac{\sqrt{2}-\sqrt{6}}{\cancel{4}}} = \frac{\sqrt{2}+\sqrt{6}}{\sqrt{2}-\sqrt{6}}$$

$$= \frac{\sqrt{2}+\sqrt{6}}{\sqrt{2}-\sqrt{6}} \cdot \frac{\sqrt{2}+\sqrt{6}}{\sqrt{2}+\sqrt{6}} = \frac{2 + 2\sqrt{12} + 6}{(\sqrt{2})^2 - (\sqrt{6})^2}$$

$$= \frac{8 + 2\sqrt{4}\sqrt{3}}{2 - 6} = \frac{8 + 4\sqrt{3}}{-4} = \frac{\cancel{4}(2+\sqrt{3})}{-\cancel{4}} = \boxed{-(2+\sqrt{3})}$$

✓

Method II $\tan 105° =$

$$\tan(\underset{A}{45°} + \underset{B}{60°}) = \frac{\tan 45° + \tan 60°}{1 - \tan 45° \tan 60°}$$

$$= \frac{1 + \sqrt{3}}{1 - \sqrt{3}}$$

$$= \frac{1+\sqrt{3}}{1-\sqrt{3}} \cdot \frac{1+\sqrt{3}}{1+\sqrt{3}} = \frac{1 + 2\sqrt{3} + 3}{(1)^2 - (\sqrt{3})^2}$$

$$= \frac{4 + 2\sqrt{3}}{1 - 3} = \frac{\cancel{2}(2+\sqrt{3})}{-\cancel{2}}$$

$$= \boxed{-(2+\sqrt{3})}$$

$\sin(A - B) = \sin(A + (-B))$
$= \sin A \cos(-B) + \cos A \sin(-B)$
$= \sin A \cos B + \cos A \cdot -\sin B$
$= \boxed{\sin A \cos B - \cos A \sin B}$

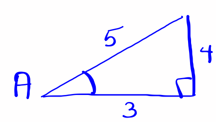
$\cos(A - B) = \cos(A + (-B))$
$= \cos A \cos(-B) - \sin A \sin(-B)$
$= \cos A \cos B - \sin A \cdot -\sin B$
$= \boxed{\cos A \cos B + \sin A \sin B}$

$\tan(A - B) = \tan(A + (-B))$
$= \dfrac{\tan A + \tan(-B)}{1 - \tan A \tan(-B)} = \boxed{\dfrac{\tan A - \tan B}{1 + \tan A \tan B}}$

Recall $\boxed{\begin{array}{l}\sin(-\alpha) = -\sin\alpha \\ \cos(-\alpha) = \cos\alpha \\ \tan(-\alpha) = -\tan\alpha\end{array}}$

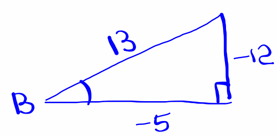
$0 < A < \frac{\pi}{2}$ $\tan A = \frac{4}{3}$

QI



$\pi < B < \frac{3\pi}{2}$ $\tan B = \frac{12}{5}$

QIII



$\sin(-B) = -\sin B$
$= -\frac{-12}{13}$
$= \boxed{\frac{12}{13}}$

$\cos(-B) = \cos B$
$= \boxed{\frac{-5}{13}}$

$\tan(-B)$
$= -\tan B$
$= \boxed{-\frac{12}{5}}$

$\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B} = \dfrac{\frac{4}{3} - \frac{12}{5}}{1 + \frac{4}{3} \cdot \frac{12}{5}}$

LCD = 15 $= \dfrac{20 - 36}{15 - 48} = \dfrac{-16}{-33}$

$= \boxed{\dfrac{16}{33}}$

Find exact value of $\cos 15°$

Hint: $15° = 45° - 30°$

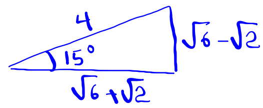
$$\cos 15° = \cos(\underset{A}{45°} - \underset{B}{30°})$$

$$= \cos 45° \cos 30° + \sin 45° \sin 30°$$

$$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \boxed{\frac{\sqrt{6}+\sqrt{2}}{4}}$$

$$\sin 15° = \sin(45° - 30°) = \sin 45° \cos 30° - \cos 45° \sin 30°$$

$$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2}$$



$$= \boxed{\frac{\sqrt{6}-\sqrt{2}}{4}}$$

verify

$$(\sqrt{6}+\sqrt{2})^2 + (\sqrt{6}-\sqrt{2})^2 = 4^2$$

$$(A+B)^2 + (A-B)^2$$

$$= A^2 + \cancel{2AB} + B^2 + A^2 - \cancel{2AB} + B^2$$

$$= 2A^2 + 2B^2 = 2(\sqrt{6})^2 + 2(\sqrt{2})^2 = 2 \cdot 6 + 2 \cdot 2 =$$

$$12 + 4 = 16$$

Derive/verify $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \cdot \tan B}$

$$\tan(A-B) = \frac{\sin(A-B)}{\cos(A-B)} = \frac{\sin A \cos B - \cos A \sin B}{\cos A \cos B + \sin A \sin B}$$

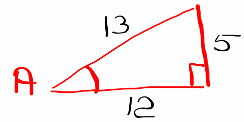
Divide everything by $\cos A \cos B$

$$= \frac{\dfrac{\sin A \cancel{\cos B}}{\cos A \cancel{\cos B}} - \dfrac{\cancel{\cos A} \sin B}{\cancel{\cos A} \cos B}}{\dfrac{\cos A \cos B}{\cos A \cos B} + \dfrac{\sin A}{\cos A}\dfrac{\sin B}{\cos B}}$$

$$= \boxed{\frac{\tan A - \tan B}{1 + \tan A \tan B}}$$

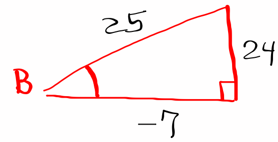
$0 < A < \frac{\pi}{2}$ QI

$\sin A = \frac{5}{13}$



$\frac{\pi}{2} < B < \pi$ QII

$\cos B = \frac{-7}{25}$



$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

$$= \frac{12}{13} \cdot \frac{-7}{25} + \frac{5}{13} \cdot \frac{24}{25}$$

$$= \frac{-84 + 120}{325} = \boxed{\frac{36}{325}}$$

QI or QIV

what quadrant is $A - B$?

$0° < A < 90°$

$90° < B < 180°$ $\rightarrow$ $-90° < A - B < -90°$?

find $\sin(A - B) = \sin A \cos B - \cos A \sin B$

$$= \frac{5}{13} \cdot \frac{-7}{25} - \frac{12}{13} \cdot \frac{24}{25}$$

$$= \frac{-35 - 288}{325} = \boxed{\frac{-323}{325}}$$

QIII or QIV

$\cos(A-B) > 0$

$\sin(A-B) < 0$ $\rightarrow$ $A - B$ is in QIV

find $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

$$= \frac{\frac{5}{12} - \frac{-24}{7}}{1 + \frac{5}{12} \cdot \frac{-24}{7}} = \frac{35 + 288}{84 - 135}$$

LCD = 12 · 7 = 84

$$= \frac{323}{-51}$$

In QIV $\rightarrow$ $\tan(A-B) < 0$

verify

I reversed it. I copied it wrong

$$\sin\left(x+\frac{\pi}{4}\right)+\sin\left(x-\frac{\pi}{4}\right)=\sqrt{2}\cos x$$

$$A+B \qquad A-B$$

$$LHS=\sin x\cos\frac{\pi}{4}+\cancel{\cos x\sin\frac{\pi}{4}}+\sin x\cos\frac{\pi}{4}-\cancel{\cos x\sin\frac{\pi}{4}}$$

$$=\sin x\cdot\frac{\sqrt{2}}{2}+\sin x\cdot\frac{\sqrt{2}}{2}=\cancel{2}\cdot\sin x\cdot\frac{\sqrt{2}}{\cancel{2}}$$

$$=\sin x\cdot\sqrt{2}$$

$$=\sqrt{2}\sin x$$

Actual Problem

$$\sin\left(x+\frac{\pi}{4}\right)+\sin\left(\frac{\pi}{4}-x\right)=\sqrt{2}\cos x$$

Verify

$$\sec(A-B)=\frac{\cos(A+B)}{\cos^2 A-\sin^2 B}$$

$$LHS=\frac{1}{\cos(A-B)}=\frac{1}{\cos A\cos B+\sin A\sin B}\cdot\frac{\cos(A+B)}{\cos(A+B)}$$

$$=\frac{1\cdot\cos(A+B)}{[\cos A\cos B+\sin A\sin B][\cos A\cos B-\sin A\sin B]}$$

$$=\frac{\cos(A+B)}{(\cos A\cos B)^2-(\sin A\sin B)^2}$$

$$=\frac{\cos(A+B)}{\cos^2 A\cos^2 B-\sin^2 A\sin^2 B}=\frac{\cos(A+B)}{\cos^2 A-\sin^2 B}$$

$$\cos^2 A(1-\sin^2 B)-(1-\cos^2 A)\sin^2 B$$

$$\cos^2 A-\cancel{\cos^2 A\sin^2 B}-\sin^2 B+\cancel{\cos^2 A\sin^2 B}$$

Verify

$$\frac{\sin(A-B)}{\cos A \cos B} = \tan A - \tan B$$

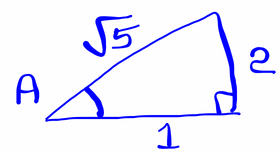
$$LHS = \frac{\sin A \cos B - \cos A \sin B}{\cos A \cos B}$$

$$= \frac{\sin A \cancel{\cos B}}{\cos A \cancel{\cos B}} - \frac{\cancel{\cos A} \sin B}{\cancel{\cos A} \cos B} = \boxed{\tan A - \tan B}$$

$\sec A = \sqrt{5}$, A is in QI.

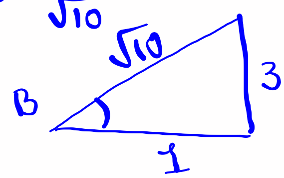
$\sec B = \sqrt{10}$, B is in QI.

find $\sec(A-B)$



$\cos A = \frac{1}{\sqrt{5}}$

$\cos B = \frac{1}{\sqrt{10}}$



$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

$$= \frac{1}{\sqrt{5}} \cdot \frac{1}{\sqrt{10}} + \frac{2}{\sqrt{5}} \cdot \frac{3}{\sqrt{10}} = \frac{7}{\sqrt{50}}$$

$$\sec(A-B) = \frac{\sqrt{50}}{7} = \frac{\sqrt{25}\sqrt{2}}{7} = \boxed{\frac{5\sqrt{2}}{7}}$$

Simplify

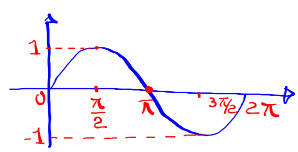
$$\cos(x+90^\circ) + \cos(x-90^\circ)$$

$$= \cos x \cos 90^\circ - \cancel{\sin x \sin 90^\circ} + \cos x \cos 90^\circ + \cancel{\sin x \sin 90^\circ}$$

$$= 2\cos x \cdot \cancelto{0}{\cos 90^\circ} = \boxed{0}$$

More on graphs

$y = \sin x$



one period is 2π

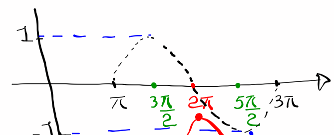
Domain $0 \le x \le 2\pi \rightarrow [0, 2\pi]$

Range $-1 \le y \le 1 \rightarrow [-1, 1]$

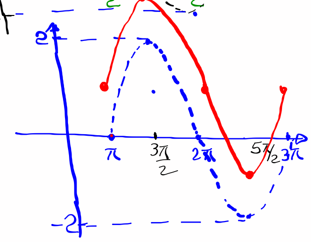
Graph $y = 2\sin(x-\pi) + 1$ Move up

1) $y = \sin(x-\pi)$ $\quad 0 \le x-\pi \le 2\pi$

$\pi \le x \le 3\pi$



2) $y = 2\sin(x-\pi)$



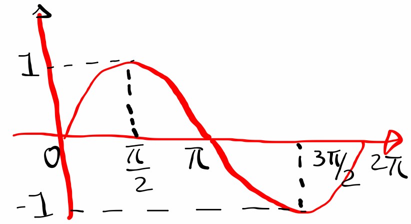
Graph $y = -\sin\left(2x + \frac{\pi}{2}\right) - 1$ → shift down

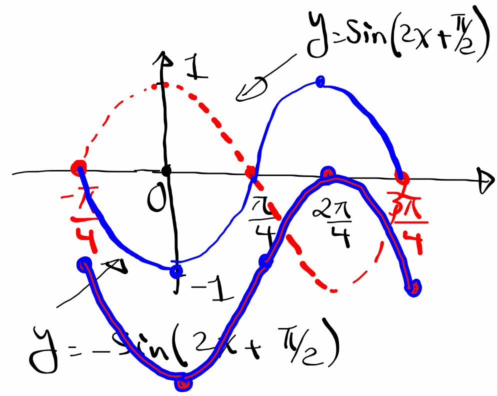
$$0 \le 2x + \frac{\pi}{2} \le 2\pi$$

$$0 \le 4x + \pi \le 4\pi$$

$$-\pi \le 4x \le 3\pi$$

$$\frac{-\pi}{4} \le x \le \frac{3\pi}{4}$$





Graph

$y = 3\sin\left(\frac{1}{2}x - \frac{\pi}{4}\right) + 5$

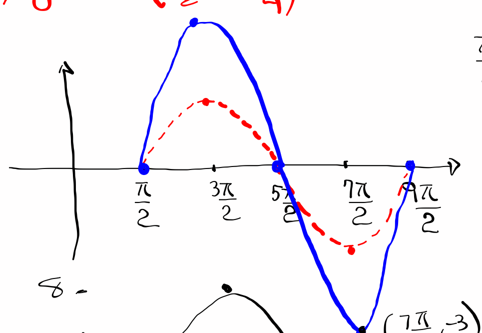
$$0 \le \frac{1}{2}x - \frac{\pi}{4} \le 2\pi$$

$$0 \le 2x - \pi \le 8\pi$$

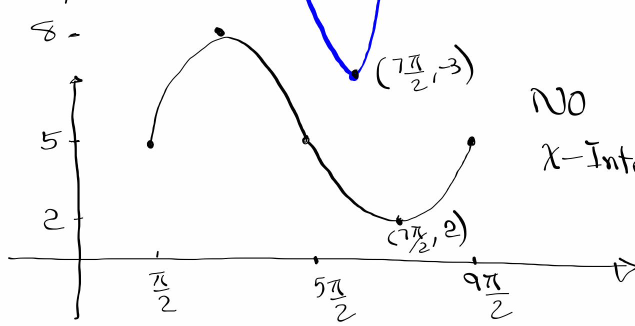
$$\pi \le 2x \le 9\pi$$

$$\frac{\pi}{2} \le x \le \frac{9\pi}{2}$$

a) $y = \sin\left(\frac{1}{2}x - \frac{\pi}{4}\right)$



b) $y = 3\sin\left(\frac{1}{2}x - \frac{\pi}{4}\right)$

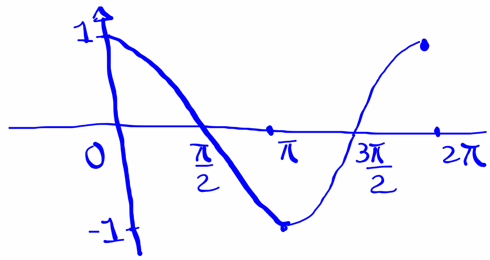


NO x-Intercepts

$y = \cos x$

Domain $[0, 2\pi]$

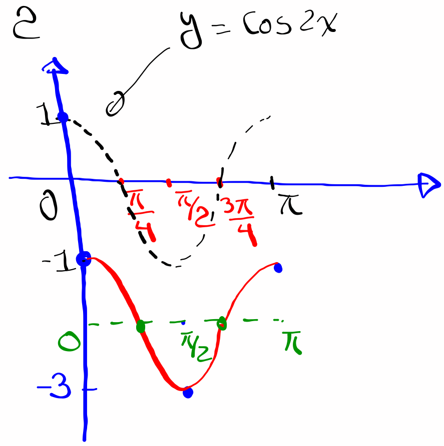
Range $[-1, 1]$



Graph $y = \cos(2x) - 2$

$0 \leq 2x \leq 2\pi$

$0 \leq x \leq \pi$



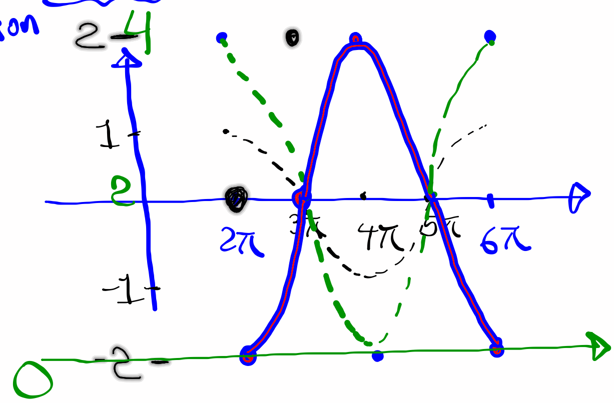
Graph $y = -2\cos\left(\frac{1}{2}x - \pi\right) + 2$

Reflection

$0 \leq \frac{1}{2}x - \pi \leq 2\pi$

$0 \leq x - 2\pi \leq 4\pi$

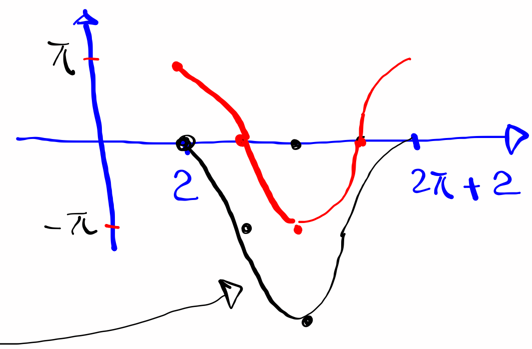
$2\pi \leq x \leq 6\pi$



Graph $y = \pi \cos(x - 2) - \pi$

$0 \le x - 2 \le 2\pi$

$2 \le x \le 2\pi + 2$



Graph $y = \cos(-x) = \cos x$

$0 \le -x \le 2\pi$

$0 \ge x \ge -2\pi$